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Some New Weighted Dynamic Inequalities for Monotone Functions Involving Kernels

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Abstract. In the present article, we derive several new weighted dynamic inequalities for monotone functions involving kernels, some of which are the Hardy-type inequalities. The established inequalities are characterized by appropriate relations for the accompanying weight functions. Our results are time scale extensions of several classical weighted inequalities known from the literature. As an application, we obtain the corresponding discrete weighted inequalities for monotone sequences, which are essentially new.

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1. Introduction

Let u and v be given weight functions, i.e., functions which are measurable and positive almost everywhere in (a, b), $-\infty \leq a < b \leq \infty$. In 1990, Opic and Kufner [12], proved that if $1 < p \leq q < \infty$, then the inequality

$$\left[\int_a^b u(x)\left(\int_a^x f(t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left(\int_a^b v(x)\, f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{1.1}$$

holds for all nonnegative measurable functions f, if and only if holds the condition

$$K = \sup_{a<x<b}\left(\int_x^b u(t)\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)\mathrm{d}t\right)^{\frac{1}{p'}} < \infty,$$

where $p' = \frac{p}{p-1}$. In addition, the estimate for the constant C appearing on the right-hand side of (1.1) is given by

$$K \leq C \leq \left(1+\frac{q}{p'}\right)^{\frac{1}{q}}\left(1+\frac{p'}{q}\right)^{\frac{1}{p'}} K. \tag{1.2}$$

The inequality (1.1) is usually referred to as the weighted Hardy inequality. The special case of $p = q$, $a = 0$, $b = \infty$ and $v(x) = 1$, $u(x) = x^{-p}$ reduces (1.1) to the classical Hardy inequality with a sharp constant $C = (p')^p$.

In 1991, Stepanov [18] proved that if $0 < p \leq 1$, $p \leq q < \infty$ and k is a nonnegative measurable kernel, then the Hardy-type inequality

$$\left[\int_0^\infty u\left(x\right)\left(\int_0^\infty k(x,y)f(y)\mathrm{d}y\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left(\int_0^\infty v\left(x\right)f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}, \quad (1.3)$$

holds for all nonnegative nondecreasing measurable functions f, if and only if

$$L = \sup_{t>0}\left(\int_t^\infty v\left(x\right)\mathrm{d}x\right)^{-\frac{1}{p}}\left[\int_0^\infty u\left(x\right)\left(\int_t^\infty k(x,y)\mathrm{d}y\right)^q \mathrm{d}x\right]^{\frac{1}{q}} < \infty.$$

In addition, if the constant C in (1.3) is the least possible, then $L = C$.

Moreover, Heinig and Maligranda [10] proved that if $0 < p \leq 1$, $p \leq q < \infty$ and k is a nonnegative measurable kernel, then the inequality

$$\left[\int_0^\infty u\left(x\right)\left(\int_0^\infty k(x,t)f(t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left(\int_0^\infty v\left(x\right)f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}, \quad (1.4)$$

holds for all nonnegative nonincreasing measurable functions f, if and only if the inequality

$$\left[\int_0^\infty u\left(x\right)\left(\int_0^s k(x,t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left(\int_0^s v\left(x\right)\mathrm{d}x\right)^{\frac{1}{p}},$$

holds for all $s > 0$. In addition, they also proved that if $0 < p \leq 1 \leq q$ and k_1, k_2 are nonnegative measurable kernels, then the inequality

$$\left[\int_0^\infty u(x)\left(\int_0^\infty k_1(x,t)f(t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left[\int_0^\infty v(x)\left(\int_0^\infty k_2(x,t)f(t)\mathrm{d}t\right)^p \mathrm{d}x\right]^{\frac{1}{p}}, \quad (1.5)$$

holds for all nonnegative nonincreasing measurable functions f and a constant $C > 0$, if and only if the relation

$$\left[\int_0^\infty u(x)\left(\int_0^s k_1(x,t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\frac{1}{q}} \leq C\left[\int_0^\infty v(x)\left(\int_0^s k_2(x,t)\mathrm{d}t\right)^p \mathrm{d}x\right]^{\frac{1}{p}},$$

holds for all $s > 0$. In 2000, Barza et al. [3] proved that if $0 < p \leq q < \infty$, $1 \leq q < \infty$ and k is a nonnegative measurable kernel, then the inequality

$$\left[\int_0^\infty u(x)f^q(x)\mathrm{d}x\right]^{\frac{1}{q}} \leq C\left[\int_0^\infty v(x)\left(\int_0^\infty k(x,y)f(y)\mathrm{d}y\right)^p \mathrm{d}x\right]^{\frac{1}{p}}, \quad (1.6)$$

holds for all nonnegative nonincreasing measurable functions f, where the constant C is defined by

$$C = \sup_{t>0}\left[\int_0^t u(x)\mathrm{d}x\right]^{\frac{1}{q}}\left[\int_0^\infty v(x)\left(\int_0^t k(x,y)\mathrm{d}y\right)^p \mathrm{d}x\right]^{-\frac{1}{p}} < \infty.$$

For the reader's convenience, the above integral inequalities (1.3)–(1.6) will be referred to as the weighted inequalities with kernels.

In the last few decades, numerous authors have been interested in establishing the corresponding discrete analogs of $L^p(\mathbb{R})$ -bounds in various fields of analysis, and as a result, this subject became topic of ongoing research. The crucial reason for this upsurge of interest in discrete case is due to the fact that discrete operators may even behave differently from their continuous counterparts. But the main challenge in establishing discrete analogs is that there are no general methods to study these questions on $l^p(\mathbb{N})$. Therefore, these methods have to be developed starting from the basic definitions. In some cases, it is possible, almost straightforward, to translate or adapt the objects and results from the continuous setting to the discrete setting or vice versa. However, in some other cases, this problem is far from being trivial, and l^p-bounds for discrete analogs of more complicated operators such as singular and fractional operators, maximal Radon transforms (involving integration over a submanifold, or family of submanifolds), are not implied by the corresponding continuous results, and moreover, they are resistant to conventional methods. In particular, related to our previous discussion about the weighted Hardy-type inequalities with kernels, Bennett and Grosse-Erdmann [4], established the discrete version of relation (1.4). More precisely, they proved that if $0 < p \leq q < \infty$, $p \leq 1$ and $(a_{n,k})$, (v_n) are nonnegative sequences, then the inequality

$$\left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{n,k}x_k\right)^q\right]^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty} v_n x_n^p\right)^{\frac{1}{p}}, \tag{1.7}$$

holds for all nonnegative nonincreasing sequences (x_n), if and only if

$$\left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m} a_{n,k}\right)^q\right]^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{m} v_n\right)^{\frac{1}{p}}, \quad \text{for all } m \in \mathbb{N},$$

where the constant C does not depend on a sequence (x_n).

On the other hand, in the recent years, the study of dynamic inequalities on time scales has received a lot of attention and became an interesting topic in pure and applied mathematics. The general idea is to establish the corresponding dynamic inequality, where the domain of an unknown function is the so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. These dynamic inequalities cover the classical continuous and discrete inequalities as special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. Besides, some new classes of inequalities are also obtained for various time scales such as $\mathbb{T} = h\mathbb{N}$, $h > 0$, $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, etc. For a comprehensive inspection of recent results on dynamic inequalities on time scales, the reader is referred to monographs [1,2,7], the papers [5,6,9,11,14–16], and references therein.

The main objective of the present paper is a study of the weighted inequalities with kernels on time scales, some of which are of the Hardy type (see, e.g., [2,11,13–17] and references therein). In particular, Saker et al. [15], established the time scale version of the weighted Hardy-type inequality (1.1), obtained by Opic and Kufner [12], which asserts that the dynamic inequality

$$\left[\int_a^b u(x)\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}} \leq C\left(\int_a^b v(x)\, f^p(x)\Delta x\right)^{\frac{1}{p}}, \tag{1.8}$$

where $1 < p \leq q < \infty$, holds for all nonnegative rd-continuous functions f on $[a,b]_{\mathbb{T}}$, a, $b \in \mathbb{T}$, if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)\Delta t\right)^{\frac{1}{q}}\left(\int_a^{\sigma(x)} v^{1-p'}(t)\Delta t\right)^{\frac{1}{p'}} = K < \infty, \quad p' = \frac{p}{p-1}. \tag{1.9}$$

Moreover, the estimate for the constant C appearing on the right-hand side of (1.8) is given by (1.2). It should be noticed here that the condition (1.9) provides a characterization of the weight functions u and v for which the inequality (1.8) holds.

The natural question that arises from the previous discussion is whether it is possible to establish some new characterizations of the corresponding weight functions, so that dynamic versions of the weighted inequalities (1.5) and (1.6) hold. In the present paper, our aim is to give an answer to this question. In particular, we will establish a relation between the weight functions λ and φ (nonnegative rd-continuous functions defined on $[a,\infty)_{\mathbb{T}}$), which will ensure that the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}} \leq C\left[\int_a^\infty \varphi(x)\left(\int_a^\infty k_2(x,t)f(t)\Delta t\right)^p \Delta x\right]^{\frac{1}{p}},$$

holds for all nonnegative nonincreasing rd-continuous functions f, provided that $0 < p \leq 1 \leq q$ and k_1, k_2 are nonnegative kernels defined on $[a,\infty)_{\mathbb{T}} \times [a,\infty)_{\mathbb{T}}$. In addition, we will also establish a relation between the weight functions λ and φ, such that the inequality

$$\left[\int_a^\infty \lambda(x)f^q(x)\Delta x\right]^{\frac{1}{q}} \leq C\left(\int_a^\infty \varphi(\tau)\left[\int_a^\infty k(\tau,x)f(x)\Delta x\right]^p \Delta\tau\right)^{\frac{1}{p}},$$

holds for all nonnegative nonincreasing rd-continuous functions f, provided that $0 < p \leq q < \infty$, $1 \leq q < \infty$, and k is a nonnegative kernel defined on $[a,\infty)_{\mathbb{T}} \times [a,\infty)_{\mathbb{T}}$.

The paper is organized as follows. After this introductory part, in Sect. 2, we present some basic definitions and facts in the theory of time scales, and prove basic lemmas that will be needed in the proofs of our main results. In Sect. 3, we prove our main results, i.e., we establish several new dynamic inequalities with kernels for a class of monotone functions. If $\mathbb{T} = \mathbb{R}$, our results provide characterizations of inequalities (1.5) and (1.6) proved by Heinig and Maligranda [10], and Barza et al. [3]. On the other hand, if $\mathbb{T} = \mathbb{N}$, it turns out that our results are essentially new.

2. Preliminaries and Basic Lemmas

In this section, we present some basic notation, definitions and properties concerning the calculus on time scales, and for more details, the reader is referred to monographs [7,8] by Bohner and Peterson, which provide a comprehensive insight into the time scale calculus. In addition, we also give several basic lemmas that will be used in the proofs of our main results.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and we define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The point t is said to be right-scattered if $\sigma(t) > t$, respectively, left-scattered if $\rho(t) < t$. The point t is called right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, respectively, left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. For an arbitrary function $f : \mathbb{T} \to \mathbb{R}$, $f^\sigma(t)$ stands for a composition $f(\sigma(t))$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at each right-dense point and if its left-sided limits exist at each left-dense point in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. We will also utilize the product and quotient rules for delta derivative f^Δ of the function f (for more details, see [7]). Namely, if f and g are delta differentiable functions on $\mathbb{T}$, then the product fg is delta differentiable on $\mathbb{T}$. Moreover, if $gg^\sigma \neq 0$, then the quotient f/g is also delta differentiable on $\mathbb{T}$, so we have

$$(fg)^\Delta = fg^\Delta + f^\Delta g^\sigma = f^\Delta g + f^\sigma g^\Delta \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{g\, g^\sigma}.$$

Throughout this paper, we deal with a delta integral which can be defined as follows: if $G^\Delta(t) = g(t)$, then the Cauchy delta integral of g is defined by $\int_a^t g(x)\Delta x := G(t) - G(a)$. It can be shown that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(x)\Delta x$, $t_0 \in \mathbb{T}$, exists and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$ (see [7]). An infinite integral is defined as $\int_a^\infty f(x)\Delta x := \lim_{b\to\infty} \int_a^b f(x)\Delta x$, while the integration on discrete time scales is given by $\int_a^b g(t)\Delta t = \sum_{t\in[a,b)} \mu(t)g(t)$. In the case when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t$, $\mu(t) = 0$, $f^\Delta = f'$, and $\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,\mathrm{d}t$, while for $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t+1$, $\rho(t) = t-1$, $\mu(t) = 1$, $f^\Delta = \Delta f$, and $\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t)$. Further, the integration by parts formula on time scales reads

$$\int_a^b u^\Delta(t)v^\sigma(t)\ \Delta t = u(t)v(t)|_a^b - \int_a^b u(t)v^\Delta(t)\Delta t. \tag{2.1}$$

We will utilize the well-known Hölder inequality in the time scale setting, which asserts that if $1/\gamma + 1/\nu = 1$, $\gamma > 1$, then

$$\int_a^b f(t)g(t)\Delta t \leq \left(\int_a^b f^\gamma(t)\Delta t\right)^{\frac{1}{\gamma}} \left(\int_a^b g^\nu(t)\Delta t\right)^{\frac{1}{\nu}}, \tag{2.2}$$

holds for $f,\, g\in C_{rd}\left([a,b]_{\mathbb{T}},\mathbb{R}^{+}\right)$. The inequality (2.2) is reversed if $0<\gamma<1$ or $\gamma<0$.

Finally, in the proofs of our results, we will also utilize a time scale version of the Fubini theorem. Let $(\Omega,\mathcal{M},\mu_{\Delta})$ and $(\Lambda,\mathcal{L},\lambda_{\Delta})$ be finite-dimensional time scale measure spaces. We define the product measure space $(\Omega\times\Lambda,\mathcal{M}\times\mathcal{L},\mu_{\Delta}\times\lambda_{\Delta})$, where $\mathcal{M}\times\mathcal{L}$ is the product $\sigma-$algebra generated by $\{E\times F: E\in\mathcal{M},F\in\mathcal{L}\}$ and $\left(\mu_{\Delta}\times\lambda_{\Delta}\right)(E\times F)=\mu_{\Delta}(E)\lambda_{\Delta}(F)$.

Theorem 2.1. [5, Theorem 1.1] *If $f:\Omega\times\Lambda\rightarrow\mathbb{R}$ is $\mu_{\Delta}\times\lambda_{\Delta}$ -integrable function and if $\varphi(y)=\int_{\Omega}f(x,y)\Delta x$ for a.e. $y\in\Lambda$, $\psi(x)=\int_{\Lambda}f(x,y)\Delta y$ for a.e. $x\in\Omega$, then φ is λ_{Δ}-integrable on Λ, ψ is μ_{Δ}-integrable on Ω, and*

$$\int_{\Omega}\Delta x\int_{\Lambda}f(x,y)\Delta y=\int_{\Lambda}\Delta y\int_{\Omega}f(x,y)\Delta x. \tag{2.3}$$

Now, we prove some basic lemmas that will be utilized in establishing our main results. If nothing else is explicitly stated, we assume that all functions are nonnegative, rd-continuous, delta differentiable and delta integrable on $[a,\infty)_{\mathbb{T}}$. Furthermore, the integrals considered are assumed to be convergent.

Lemma 2.1. *Let $\mathbb{T}$ be a time scale with $a\in\mathbb{T}$, and let f and ϕ be nonnegative rd-continuous functions defined on $[a,\infty)_{\mathbb{T}}$. If $\lim_{t\rightarrow\infty}f(t)=0$, then holds the relation*

$$\int_{a}^{\infty}[-f^{\Delta}(t)]\left(\int_{a}^{\sigma(t)}\phi(\tau)\Delta\tau\right)\Delta t=\int_{a}^{\infty}f(t)\phi(t)\Delta t. \tag{2.4}$$

Proof. Applying the integration by parts to the term

$$\int_{a}^{\infty}[-f^{\Delta}(t)]\left(\int_{a}^{\sigma(t)}\phi(\tau)\Delta\tau\right)\Delta t,$$

with

$$u^{\Delta}(t)=-f^{\Delta}(t)\text{ and }v^{\sigma}(t)=\int_{a}^{\sigma(t)}\phi(\tau)\Delta\tau=\Phi^{\sigma}(t),$$

we get

$$\int_{a}^{\infty}[-f^{\Delta}(t)]\left(\int_{a}^{\sigma(t)}\phi(\tau)\Delta\tau\right)\Delta t=-f(t)\Phi(t)|_{a}^{\infty}+\int_{a}^{\infty}f(t)\Phi^{\Delta}(t)\Delta t,$$

where $\Phi(t)=\int_{a}^{t}\phi(\tau)\Delta\tau$. Now, since $\lim_{t\rightarrow\infty}f(t)=0$ and $\Phi(a)=0$, it follows that

$$\int_{a}^{\infty}[-f^{\Delta}(t)]\left(\int_{a}^{\sigma(t)}\phi(\tau)\Delta\tau\right)\Delta t=\int_{a}^{\infty}f(t)\Phi^{\Delta}(t)\Delta t=\int_{a}^{\infty}f(t)\phi(t)\Delta t,$$

which represents the desired inequality (2.4). □

Lemma 2.2. *Assume that* $\mathbb{T}$ *is a time scale with* $a \in \mathbb{T}$, *and let* f *and* ϕ *be nonnegative rd-continuous functions defined on* $[a, \infty)_{\mathbb{T}}$. *If the function* f *is bounded and* $f(a) = 0$, *then holds the relation*

$$\int_a^\infty f^\Delta(t) \left(\int_t^\infty \phi(\tau)\Delta\tau \right) \Delta t = \int_a^\infty \phi(t) f^\sigma(t)\Delta t. \tag{2.5}$$

Proof. Integrating by parts the term

$$\int_a^\infty f^\Delta(t) \left(\int_t^\infty \phi(\tau)\Delta\tau \right) \Delta t,$$

with $u(t) = \int_t^\infty \phi(\tau)\Delta\tau$ and $v^\Delta(t) = f^\Delta(t)$, it follows that

$$\int_a^\infty f^\Delta(t) \left(\int_t^\infty \phi(\tau)\Delta\tau \right) \Delta t = u(t)f(t)|_a^\infty - \int_a^\infty u^\Delta(t) f^\sigma(t)\Delta t.$$

Now, since the function f is bounded, taking into account that $\lim_{t\to\infty} u(t) = 0$ and $f(a) = 0$, we have

$$\int_a^\infty f^\Delta(t) \left(\int_t^\infty \phi(\tau)\Delta\tau \right) \Delta t = -\int_a^\infty u^\Delta(t) f^\sigma(t)\Delta t = \int_a^\infty \phi(t) f^\sigma(t)\Delta t,$$

which yields relation (2.5). The proof is complete. □

The previous two lemmas will be utilized in establishing time scale extensions of the inequality (1.5). On the other hand, the following inequality will be the crucial step in deriving a time scale version of the inequality (1.6).

Lemma 2.3. *Let* $\mathbb{T}$ *is a time scale with* $a \in \mathbb{T}$, *and let* $\alpha, \beta \in C_{rd}([a, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. *If* $\gamma \geq 1$, *then holds the inequality*

$$\left[\int_a^\infty \alpha(x) \left(\int_a^{\sigma(x)} \beta(\tau)\Delta\tau \right)^\gamma \Delta x \right]^{\frac{1}{\gamma}} \leq \int_a^\infty \beta(x) \left(\int_x^\infty \alpha(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta x. \tag{2.6}$$

Proof. Let $F(x) = \int_a^x \beta(\tau)\Delta\tau$. Then, the left-hand side of (2.6) can be rewritten in the following form:

$$\int_a^\infty \alpha(x) \left(\int_a^{\sigma(x)} \beta(\tau)\Delta\tau \right)^\gamma \Delta x = \int_a^\infty \alpha(x) \left[F^\sigma(x)\right]^{\gamma-1} F^\sigma(x)\Delta x. \tag{2.7}$$

Furthermore, integrating by parts the right-hand side of (2.7) with

$$u^\Delta(x) = \alpha(x) \left[F^\sigma(x)\right]^{\gamma-1} \quad \text{and} \quad v^\sigma(x) = F^\sigma(x),$$

it follows that

$$\int_a^\infty \alpha(x) \left[F^\sigma(x)\right]^{\gamma-1} F^\sigma(x)\Delta x = u(x)F(x)|_a^\infty - \int_a^\infty u(x)\beta(x)\Delta x,$$

where $u(x) = -\int_x^\infty \alpha(s) \left[F^\sigma(s)\right]^{\gamma-1} \Delta s$. Clearly, since $\lim_{x\to\infty} u(x) = 0$ and $F(a) = 0$, we obtain the relation

$$\int_a^\infty \alpha(x) \left[F^\sigma(x)\right]^{\gamma-1} F^\sigma(x)\Delta x = \int_a^\infty \beta(x) \left(\int_x^\infty \alpha(s) \left[F^\sigma(s)\right]^{\gamma-1} \Delta s \right) \Delta x. \tag{2.8}$$

Now, taking into account (2.7) and (2.8), we have

$$\begin{aligned}
&\int_a^\infty \alpha(x)\left(\int_a^{\sigma(x)}\beta(\tau)\Delta\tau\right)^\gamma \Delta x\\
&\quad=\int_a^\infty \beta(x)\left(\int_x^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma-1}\Delta s\right)\Delta x\\
&\quad=\int_a^\infty \beta(x)\left(\int_x^\infty \alpha^{\frac{1}{\gamma}}(s)\alpha^{\frac{\gamma-1}{\gamma}}(s)\left[F^\sigma(s)\right]^{\gamma-1}\Delta s\right)\Delta x. \qquad (2.9)
\end{aligned}$$

On the other hand, applying the Hölder inequality with nonnegative exponents γ and $\gamma/(\gamma-1)$ to the term

$$\int_x^\infty \alpha^{\frac{1}{\gamma}}(s)\alpha^{\frac{\gamma-1}{\gamma}}(s)\left[F^\sigma(s)\right]^{\gamma-1}\Delta s,$$

we obtain the inequality

$$\begin{aligned}
&\int_x^\infty \alpha^{\frac{1}{\gamma}}(s)\alpha^{\frac{\gamma-1}{\gamma}}(s)\left[F^\sigma(s)\right]^{\gamma-1}\Delta s\\
&\quad\le\left(\int_x^\infty \alpha(s)\Delta s\right)^{\frac{1}{\gamma}}\left(\int_x^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s\right)^{\frac{\gamma-1}{\gamma}}. \qquad (2.10)
\end{aligned}$$

Now, comparing relations (2.9) and (2.10), we obtain the inequality

$$\begin{aligned}
&\int_a^\infty \alpha(x)\left(\int_a^{\sigma(x)}\beta(\tau)\Delta\tau\right)^\gamma \Delta x\\
&\quad\le\int_a^\infty \beta(x)\left(\int_x^\infty \alpha(s)\Delta s\right)^{\frac{1}{\gamma}}\left(\int_x^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s\right)^{\frac{\gamma-1}{\gamma}}\Delta x. \qquad (2.11)
\end{aligned}$$

Moreover, since $x \ge a$ and α, β are nonnegative functions, it follows that

$$\int_x^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s \le \int_a^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s,$$

and consequently,

$$\left(\int_x^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s\right)^{\frac{\gamma-1}{\gamma}} \le \left(\int_a^\infty \alpha(s)\left[F^\sigma(s)\right]^{\gamma}\Delta s\right)^{\frac{\gamma-1}{\gamma}}, \qquad (2.12)$$

since the exponent $(\gamma-1)/\gamma$ is positive. Finally, utilizing relations (2.11) and (2.12), we obtain the inequality

$$\int_a^{\infty} \alpha(x) \left(\int_a^{\sigma(x)} \beta(\tau)\Delta\tau \right)^{\gamma} \Delta x$$
$$\leq \left(\int_a^{\infty} \alpha(s) \left[F^{\sigma}(s) \right]^{\gamma} \Delta s \right)^{\frac{\gamma-1}{\gamma}} \int_a^{\infty} \beta(x) \left(\int_x^{\infty} \alpha(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta x$$
$$= \left[\int_a^{\infty} \alpha(s) \left(\int_a^{\sigma(s)} \beta(\tau)\Delta\tau \right)^{\gamma} \Delta s \right]^{\frac{\gamma-1}{\gamma}} \int_a^{\infty} \beta(x) \left(\int_x^{\infty} \alpha(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta x,$$

and so,

$$\left[\int_a^{\infty} \alpha(x) \left(\int_a^{\sigma(x)} \beta(\tau)\Delta\tau \right)^{\gamma} \Delta x \right]^{\frac{1}{\gamma}} \leq \int_a^{\infty} \beta(x) \left(\int_x^{\infty} \alpha(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta x,$$

which represents (2.6). The proof is now complete. □

It should be noticed here that the relation (2.6) is a Minkowski-type inequality in a time scale setting. Following the same lines as in the proof of Lemma 2.3, we also obtain the Minkowski-type inequality which is, in some way, complementary to (2.6).

Lemma 2.4. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, and let α, $\beta \in C_{rd}\left([a,\infty)_{\mathbb{T}}, \mathbb{R}^{+}\right)$. If $\gamma \geq 1$, then holds the inequality*

$$\left[\int_a^{\infty} \alpha(x) \left(\int_x^{\infty} \beta(t)\Delta t \right)^{\gamma} \Delta x \right]^{\frac{1}{\gamma}} \leq \int_a^{\infty} \beta(x) \left(\int_a^{\sigma(x)} \alpha(s)\Delta s \right)^{\frac{1}{\gamma}} \Delta x. \tag{2.13}$$

Our last result in this section is a Minkowski-type inequality involving a nonnegative kernel, which will be utilized in extending both relations (1.5) and (1.6) to a time scale setting.

Lemma 2.5. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and let $\gamma \geq 1$. If $k : [a,\infty)_{\mathbb{T}} \times [a,\infty)_{\mathbb{T}} \to \mathbb{R}$, $w, h : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ are nonnegative rd-continuous functions, then holds the inequality*

$$\left[\int_a^{\infty} w(x) \left(\int_a^{\infty} h(t)k(x,t)\Delta t \right)^{\gamma} \Delta x \right]^{\frac{1}{\gamma}}$$
$$\leq \int_a^{\infty} h(t) \left(\int_a^{\infty} w(x)k^{\gamma}(x,t)\Delta x \right)^{\frac{1}{\gamma}} \Delta t. \tag{2.14}$$

Proof. Let H be an integral operator defined by

$$H(x) := \int_a^{\infty} h(t)k(x,t)\Delta t. \tag{2.15}$$

Then, the left-hand side of the inequality (2.14) can be rewritten in the following form:

$$\int_a^{\infty} w(x) \left[\int_a^{\infty} h(t)k(x,t)\Delta t \right]^{\gamma} \Delta x = \int_a^{\infty} w(x)H^{\gamma-1}(x)H(x)\Delta x$$

$$= \int_a^\infty w(x)H^{\gamma-1}(x)\left(\int_a^\infty h(t)k(x,t)\Delta t\right)\Delta x$$
$$= \int_a^\infty \left(\int_a^\infty w(x)H^{\gamma-1}(x)h(t)k(x,t)\Delta t\right)\Delta x. \tag{2.16}$$

Moreover, applying the Fubini theorem to the right-hand side of (2.16), it follows that

$$\int_a^\infty \left(\int_a^\infty w(x)H^{\gamma-1}(x)h(t)k(x,t)\Delta t\right)\Delta x$$
$$= \int_a^\infty h(t)\left(\int_a^\infty w(x)H^{\gamma-1}(x)k(x,t)\Delta x\right)\Delta t$$
$$= \int_a^\infty h(t)\left(\int_a^\infty w^{\frac{1}{\gamma}}(x)k(x,t)w^{\frac{\gamma-1}{\gamma}}(x)H^{\gamma-1}(x)\Delta x\right)\Delta t. \tag{2.17}$$

Clearly, from (2.16) and (2.17), we obtain the relation

$$\int_a^\infty w(x)\left[\int_a^\infty h(t)k(x,t)\Delta t\right]^\gamma \Delta x$$
$$= \int_a^\infty h(t)\left(\int_a^\infty w^{\frac{1}{\gamma}}(x)k(x,t)w^{\frac{\gamma-1}{\gamma}}(x)H^{\gamma-1}(x)\Delta x\right)\Delta t. \tag{2.18}$$

Now, applying the Hölder inequality with positive exponents γ and $\gamma/(\gamma-1)$, to the term

$$\int_a^\infty w^{\frac{1}{\gamma}}(x)k(x,t)w^{\frac{\gamma-1}{\gamma}}(x)H^{\gamma-1}(x)\Delta x,$$

we obtain the inequality

$$\int_a^\infty w^{\frac{1}{\gamma}}(x)k(x,t)w^{\frac{\gamma-1}{\gamma}}(x)H^{\gamma-1}(x)\Delta x$$
$$\le \left(\int_a^\infty w(x)k^\gamma(x,t)\Delta x\right)^{\frac{1}{\gamma}}\left(\int_a^\infty w(x)H^\gamma(x)\Delta x\right)^{\frac{\gamma-1}{\gamma}}. \tag{2.19}$$

Hence, comparing relations (2.18) and (2.19), it follows that

$$\int_a^\infty w(x)\left[\int_a^\infty h(t)k(x,t)\Delta t\right]^\gamma \Delta x$$
$$\le \left[\int_a^\infty h(t)\left(\int_a^\infty w(x)k^\gamma(x,t)\Delta x\right)^{\frac{1}{\gamma}}\Delta t\right]\left(\int_a^\infty w(x)H^\gamma(x)\Delta x\right)^{\frac{\gamma-1}{\gamma}} \tag{2.20}$$

Finally, taking into account definition (2.15), the inequality (2.20) reduces to

$$\left(\int_a^\infty w(x)\left[\int_a^\infty h(t)k(x,t)\Delta t\right]^\gamma \Delta x\right)^{\frac{1}{\gamma}} \le \int_a^\infty h(t)\left(\int_a^\infty w(x)k^\gamma(x,t)\Delta x\right)^{\frac{1}{\gamma}}\Delta t,$$

i.e., we obtain (2.14), as claimed. The proof is complete. □

3. Main Results

In this section, we state and prove our main results, i.e., we establish time scale versions of inequalities (1.5) and (1.6), presented in the introduction. Similar to the classical real case, we will show that these inequalities in the time scale setting are also characterized by appropriate relations for the corresponding weight functions. Moreover, the results that follow will be established for a class of monotone functions (nonincreasing or nondecreasing), to preserve the sign of the corresponding inequality.

To simplify our further discussion, we assume that all functions are rd-continuous, delta differentiable and delta integrable on $[a,\infty)_{\mathbb{T}}$; so, these types of conditions will be omitted.

Our first result is a time scale version of relation (1.5), which is characterized in the same way as in the classical real case.

Theorem 3.1. *Let* $\mathbb{T}$ *be a time scale with* $a \in \mathbb{T}$*, let* $0 < p \le 1 \le q$*, and let* $\lambda, \varphi : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$*,* $k_1, k_2 : [a,\infty)_{\mathbb{T}} \times [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative functions. If* $f : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ *is a nonnegative nonincreasing function such that* $\lim_{s\to\infty} f(s) = 0$*, then the inequality*

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}} \le C\left[\int_a^\infty \varphi(x)\left(\int_a^\infty k_2(x,t)f(t)\Delta t\right)^p \Delta x\right]^{\frac{1}{p}}, \tag{3.1}$$

holds if and only if the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)^q \Delta x\right]^{\frac{1}{q}} \le C\left[\int_a^\infty \varphi(x)\left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)^p \Delta x\right]^{\frac{1}{p}}, \tag{3.2}$$

holds for all $t \in [a,\infty)_{\mathbb{T}}$*.*

Proof. The necessity part of Theorem follows by taking

$$f(t) = \begin{cases} 1, & t \in [a,\sigma(s)]_{\mathbb{T}} \\ 0, & \text{otherwise,} \end{cases}$$

for any fixed number $s \in [a,\infty)_{\mathbb{T}}$.

Now, suppose that the inequality (3.2) is valid. Applying Lemma 2.1 with $\phi(t) = k_1(x,t)$, it follows that

$$\int_a^\infty [-f^\Delta(t)]\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)\Delta t = \int_a^\infty f(t)k_1(x,t)\Delta t.$$

Therefore, the left-hand side of the inequality (3.1) can be rewritten as

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$= \left[\int_a^\infty \lambda(x)\left(\int_a^\infty [-f^\Delta(t)]\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}. \tag{3.3}$$

On the other hand, utilizing relation (2.14) with

$$\gamma = q,\ w(x) = \lambda(x),\ h(t) = [-f^\Delta(t)] \text{ and } k(x,t) = \int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau,$$

we obtain the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty [-f^\Delta(t)]\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$\leq \int_a^\infty [-f^\Delta(t)]\left[\int_a^\infty \lambda(x)\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)^q \Delta x\right]^{\frac{1}{q}} \Delta t. \tag{3.4}$$

Hence, relations (3.3) and (3.4) yield the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$\leq \int_a^\infty [-f^\Delta(t)]\left[\int_a^\infty \lambda(x)\left(\int_a^{\sigma(t)} k_1(x,\tau)\Delta\tau\right)^q \Delta x\right]^{\frac{1}{q}} \Delta t. \tag{3.5}$$

Now, combining the inequalities (3.2) and (3.5), and utilizing the fact that f is nonincreasing function, we obtain the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$\leq C\int_a^\infty [-f^\Delta(t)]\left[\int_a^\infty \varphi(x)\left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)^p \Delta x\right]^{\frac{1}{p}} \Delta t. \tag{3.6}$$

Furthermore, yet another application of the inequality (2.14) with

$$\gamma = 1/p,\ w(x) = [-f^\Delta(x)],\ h(t) = \varphi(t) \quad \text{and} \quad k(x,t) = \left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)^p,$$

yields

$$\left[\int_a^\infty [-f^\Delta(t)]\left[\int_a^\infty \varphi(x)\left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)^p \Delta x\right]^{\frac{1}{p}} \Delta t\right]^p$$
$$\leq \int_a^\infty \varphi(x)\left(\int_a^\infty [-f^\Delta(t)]\left[\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right]\Delta t\right)^p \Delta x,$$

and so

$$\int_a^\infty [-f^\Delta(t)]\left[\int_a^\infty \varphi(x)\left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)^p \Delta x\right]^{\frac{1}{p}} \Delta t$$
$$\le \left[\int_a^\infty \varphi(x)\left(\int_a^\infty [-f^\Delta(t)]\left[\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right]\Delta t\right)^p \Delta x\right]^{\frac{1}{p}}. \tag{3.7}$$

Clearly, combining relations (3.6) and (3.7), we arrive to the inequality

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$\le C\left(\int_a^\infty \varphi(x)\left(\int_a^\infty [-f^\Delta(t)]\left[\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right]\Delta t\right)^p \Delta x\right)^{\frac{1}{p}}. \tag{3.8}$$

Finally, yet another application of Lemma 2.1, this time with $\phi(t) = k_2(x,t)$, yields relation

$$\int_a^\infty [-f^\Delta(t)]\left(\int_a^{\sigma(t)} k_2(x,\tau)\Delta\tau\right)\Delta t = \int_a^\infty f(t)k_2(x,t)\Delta t,$$

so the inequality (3.8) becomes

$$\left[\int_a^\infty \lambda(x)\left(\int_a^\infty k_1(x,t)f(t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}}$$
$$\le C\left[\int_a^\infty \varphi(x)\left(\int_a^\infty k_2(x,t)f(t)\Delta t\right)^p \Delta x\right]^{\frac{1}{p}},$$

as claimed. The proof is complete. □

Remark 3.1. It should be noticed here that if $\mathbb{T} = \mathbb{R}$ and $a = 0$, then the inequality (3.1) reduces to inequality (1.5) established by Heinig and Maligranda (see also [10]). Another interesting special case of Theorem 3.1 refers to a classical discrete setting. Namely, if $\mathbb{T} = \mathbb{N}$ and $a = 1$, then $f(n)$ is a nonnegative nonincreasing sequence such that $\lim_{n\to\infty} f(n) = 0$. Therefore, if $0 < p \le 1 \le q$, then the inequality

$$\left[\sum_{n=1}^\infty \lambda(n)\left(\sum_{m=1}^\infty k_1(n,m)f(m)\right)^q\right]^{\frac{1}{q}} \le C\left[\sum_{n=1}^\infty \varphi(n)\left(\sum_{m=1}^\infty k_2(n,m)f(m)\right)^p\right]^{\frac{1}{p}},$$

holds if and only if the inequality

$$\left[\sum_{n=1}^\infty \lambda(n)\left(\sum_{m=1}^N k_1(n,m)\right)^q\right]^{\frac{1}{q}} \le C\left[\sum_{n=1}^\infty \varphi(n)\left(\sum_{m=1}^N k_2(n,m)\right)^p\right]^{\frac{1}{p}},$$

holds for all $N \in \mathbb{N}$. Here, $\lambda(n), \varphi(n), k_1(n,m), k_2(n,m)$ are nonnegative sequences of real numbers.

The following result is, in some way, complementary to Theorem 3.1, since it refers to a class of nondecreasing functions.

Theorem 3.2. *Let* $\mathbb{T}$ *be a time scale with* $a \in \mathbb{T}$, *let* $0 < p \leq 1 \leq q$, *and let* $\lambda, \varphi : [a, \infty)_{\mathbb{T}} \to \mathbb{R}$, $k_1, k_2 : [a, \infty)_{\mathbb{T}} \times [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative functions. If* $f : [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ *is a nonnegative nondecreasing bounded function such that* $f(a) = 0$, *then the inequality*

$$\left[\int_a^{\infty} \lambda(x) \left(\int_a^{\infty} k_1(x,t) f(t) \Delta t\right)^q \Delta x\right]^{\frac{1}{q}} \leq C \left[\int_a^{\infty} \varphi(x) \left(\int_a^{\infty} k_2(x,t) f^{\sigma}(t) \Delta t\right)^p \Delta x\right]^{\frac{1}{p}},$$

holds if and only if the inequality

$$\left[\int_a^{\infty} \lambda(x) \left(\int_t^{\infty} k_1(x,\tau) \Delta \tau\right)^q \Delta x\right]^{\frac{1}{q}} \leq C \left[\int_a^{\infty} \varphi(x) \left(\int_t^{\infty} k_2(x,\tau) \Delta \tau\right)^p \Delta x\right]^{\frac{1}{p}},$$

holds for all $t \in [a, \infty)_{\mathbb{T}}$.

Proof. The proof is similar to the proof of Theorem 3.1, except that we use Lemma 2.2 instead of Lemma 2.1, and the fact that f is nondecreasing function. □

Now, our intention is to establish a time scale version of the Hardy-type inequality (1.6). Similar to Theorem 3.1, the corresponding generalization will also be characterized via the appropriate inequality for the accompanying weight functions.

Theorem 3.3. *Let* $\mathbb{T}$ *be a time scale with* $a \in \mathbb{T}$, *let* $0 < p \leq q$, $1 \leq q$, *and let* $\lambda, \varphi : [a, \infty)_{\mathbb{T}} \to \mathbb{R}$, $k : [a, \infty)_{\mathbb{T}} \times [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative functions. If* $f : [a, \infty)_{\mathbb{T}} \to \mathbb{R}$ *is a nonnegative nonincreasing function such that* $\lim_{s \to \infty} f(s) = 0$, *then the inequality*

$$\left(\int_a^{\infty} \lambda(x) f^q(x) \Delta x\right)^{\frac{1}{q}} \leq C \left[\int_a^{\infty} \varphi(\tau) \left(\int_a^{\infty} k(\tau, x) f(x) \Delta x\right)^p \Delta \tau\right]^{\frac{1}{p}}, \tag{3.9}$$

holds if and only if the inequality

$$\left(\int_a^{\sigma(x)} \lambda(\tau) \Delta \tau\right)^{\frac{1}{q}} \leq C \left[\int_a^{\infty} \varphi(\tau) \left(\int_a^{\sigma(x)} k(\tau, t) \Delta t\right)^p \Delta \tau\right]^{\frac{1}{p}}, \tag{3.10}$$

holds for all $x \in [a, \infty)_{\mathbb{T}}$.

Proof. The necessity part of Theorem follows by taking

$$f(t) = \begin{cases} 1, & t \in [a, \sigma(s)]_{\mathbb{T}} \\ 0, & \text{otherwise}, \end{cases}$$

for any fixed number $s \in [a, \infty)_{\mathbb{T}}$.

To prove sufficiency, we start by integrating the term $\int_a^\infty \lambda(x)f^q(x)\Delta x$. Namely, utilizing the integration by parts formula with $u(x) = f^q(x)$ and $v^\Delta(x) = \lambda(x)$, we have

$$\int_a^\infty \lambda(x)f^q(x)\Delta x = f^q(x)v(x)|_a^\infty - \int_a^\infty [f^q(x)]^\Delta v^\sigma(x)\Delta x,$$

where $v(x) = \int_a^x \lambda(\tau)\Delta\tau$. Moreover, since $v(a) = 0$ and $\lim_{x\to\infty} f(x) = 0$, it follows that

$$\begin{aligned}\int_a^\infty \lambda(x)f^q(x)\Delta x &= \int_a^\infty [-f^q(x)]^\Delta v^\sigma(x)\Delta x\\ &= \int_a^\infty [-f^q(x)]^\Delta \left(\int_a^{\sigma(x)} \lambda(\tau)\Delta\tau\right)\Delta x. \qquad (3.11)\end{aligned}$$

Hence, combining relations (3.10) and (3.11), and utilizing the fact that f is nonincreasing function, we obtain

$$\int_a^\infty \lambda(x)f^q(x)\Delta x \le C^q \int_a^\infty [-f^q(x)]^\Delta \left[\int_a^\infty \varphi(\tau)\left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^p \Delta\tau\right]^{\frac{q}{p}} \Delta x,$$

and consequently,

$$\begin{aligned}&\left(\int_a^\infty \lambda(x)f^q(x)\Delta x\right)^{\frac{p}{q}}\\ &\quad\le C^p \left(\int_a^\infty [-f^q(x)]^\Delta \left[\int_a^\infty \varphi(\tau)\left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^p \Delta\tau\right]^{\frac{q}{p}} \Delta x\right)^{\frac{p}{q}}.\end{aligned} \qquad (3.12)$$

Now, applying Lemma 2.5 with $\gamma = q/p$, $w(x) = [-f^q(x)]^\Delta$, $h(\tau) = \varphi(\tau)$ and

$$k(x,\tau) = \left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^p,$$

to the term

$$\left(\int_a^\infty [-f^q(x)]^\Delta \left[\int_a^\infty \varphi(\tau)\left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^p \Delta\tau\right]^{\frac{q}{p}} \Delta x\right)^{\frac{p}{q}},$$

we obtain the inequality

$$\begin{aligned}&\left(\int_a^\infty [-f^q(x)]^\Delta \left[\int_a^\infty \varphi(\tau)\left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^p \Delta\tau\right]^{\frac{q}{p}} \Delta x\right)^{\frac{p}{q}}\\ &\quad\le \int_a^\infty \varphi(\tau)\left(\int_a^\infty [-f^q(x)]^\Delta \left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^q \Delta x\right)^{\frac{p}{q}} \Delta\tau.\end{aligned} \qquad (3.13)$$

Clearly, relations (3.12) and (3.13) yield the inequality

$$\left(\int_a^\infty \lambda(x) f^q(x)\Delta x\right)^{\frac{p}{q}} \leq C^p \int_a^\infty \varphi(\tau)\left(\int_a^\infty [-f^q(x)]^\Delta \left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^q \Delta x\right)^{\frac{p}{q}} \Delta\tau. \tag{3.14}$$

On the other hand, applying Lemma 2.3 to the term

$$\left[\int_a^\infty [-f^q(x)]^\Delta \left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}},$$

with $\gamma = q$, $\alpha(x) = [-f^q(x)]^\Delta$, $\beta(t) = k_2(\tau,t)$, and noting that $\lim_{s\to\infty} f(s) = 0$, we obtain the inequality

$$\left[\int_a^\infty [-f^q(x)]^\Delta \left(\int_a^{\sigma(x)} k(\tau,t)\Delta t\right)^q \Delta x\right]^{\frac{1}{q}} \leq \int_a^\infty k(\tau,x)\left(\int_x^\infty [-f^q(s)]^\Delta \Delta s\right)^{\frac{1}{q}} \Delta x = \int_a^\infty k(\tau,x) f(x)\Delta x. \tag{3.15}$$

Finally, from the last two relations (3.14) and (3.15), we obtain the inequality

$$\left(\int_a^\infty \lambda(x) f^q(x)\Delta x\right)^{\frac{p}{q}} \leq C^p \int_a^\infty \varphi(\tau)\left[\int_a^\infty k(\tau,x) f(x)\Delta x\right]^p \Delta\tau,$$

and consequently,

$$\left(\int_a^\infty \lambda(x) f^q(x)\Delta x\right)^{\frac{1}{q}} \leq C\left[\int_a^\infty \varphi(\tau)\left(\int_a^\infty k(\tau,x) f(x)\Delta x\right)^p \Delta\tau\right]^{\frac{1}{p}},$$

which proves our assertion. The proof is complete. □

Remark 3.2. If $\mathbb{T} = \mathbb{R}$ and $a = 0$, then the inequality (3.9) reduces to the inequality (1.6), established in [3]. On the other hand, for $\mathbb{T} = \mathbb{N}$ and $a = 1$, the Theorem 3.3 provides the corresponding classical discrete setting. In this case $f(n)$ is a nonnegative nonincreasing sequence such that $\lim_{n\to\infty} f(n) = 0$. Therefore, if $0 < p \leq q$ and $1 \leq q$, then the inequality

$$\left(\sum_{n=1}^\infty \lambda(n) f^q(n)\right)^{\frac{1}{q}} \leq C\left[\sum_{n=1}^\infty \varphi(n)\left(\sum_{m=1}^\infty k(n,m) f(m)\right)^p\right]^{\frac{1}{p}},$$

holds if and only if the inequality

$$\left(\sum_{n=1}^N \lambda(n)\right)^{\frac{1}{q}} \leq C\left[\sum_{n=1}^\infty \varphi(n)\left(\sum_{m=1}^N k(n,m)\right)^p\right]^{\frac{1}{p}},$$

holds for all $N \in \mathbb{N}$. Here, $\lambda(n), \varphi(n), k(n,m)$ are nonnegative sequences of real numbers.

To conclude this paper, it remains to establish a version of Theorem 3.3 that corresponds to a class of nondecreasing functions.

Theorem 3.4. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, let $0 < p \leq q$, $1 \leq q$, and let $\lambda, \varphi : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$, $k : [a,\infty)_{\mathbb{T}} \times [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative functions. If $f : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ is a nonnegative nondecreasing bounded function such that $f(a) = 0$, then the inequality*

$$\left(\int_a^\infty \lambda(x) f^q(x)\Delta x\right)^{\frac{1}{q}} \leq C\left[\int_a^\infty \varphi(\tau)\left(\int_a^\infty k(\tau,x) f^\sigma(x)\Delta x\right)^p \Delta\tau\right]^{\frac{1}{p}},$$

holds if and only if the inequality

$$\left(\int_x^\infty \lambda(\tau)\Delta\tau\right)^{\frac{1}{q}} \leq C\left[\int_a^\infty \varphi(\tau)\left(\int_x^\infty k_2(\tau,t)\Delta t\right)^p \Delta\tau\right]^{\frac{1}{p}},$$

holds for all $x \in [a,\infty)_{\mathbb{T}}$.

Proof. We follow the lines as in the proof of Theorem 3.3 except that we use Lemma 2.4 instead of Lemma 2.3, and the fact that f is nondecreasing function. □

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